

The Algebraic and Geometric Unification of E_8 Through $SU(3)$, $SU(2)$, $U(1)$, and H_4 Symmetries: A Pre-Dynamical Framework for Emergent Algebraic Invariants

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Abstract

The exceptional Lie group E_8 (rank 8, dimension 248) provides a unique unification of algebraic and geometric symmetries. It contains the classical gauge groups $SU(3)$, $SU(2)$, and $U(1)$ via maximal regular rank-preserving embeddings, while its root lattice decomposes geometrically into two golden-ratio-scaled copies of the non-crystallographic Coxeter group H_4 (symmetry of the 600-cell) through the even icosian lattice.

This paper presents a rigorous synthesis of these structures within a purely algebraic, pre-dynamical framework. We introduce the Recursion-Invariant functor (RIF) on a category of exceptional-octonionic objects, which extracts recursion-stable invariants along the maximal unifying chain

$$H_4 \longrightarrow E_8 \supset E_7 \times SU(2) \supset E_6 \times U(1) \supset SU(3)^3 \times U(1),$$

extended structurally by the full exceptional series $G_2 \rightarrow F_4 \rightarrow E_6 \rightarrow E_7 \rightarrow E_8$.

The RIF functor yields exactly seven non-trivial fundamental algebraic invariants, one per primary symmetry class in the unifying chain. Lower exceptional groups G_2 and F_4 mediate the octonionic foundation but do not introduce independent RIF-stable invariants. The exceptional-octonionic hierarchy naturally embeds the gauge-algebra structure underlying the Standard Model. No dynamical assumptions are required.

1 Introduction

The largest exceptional Lie group E_8 occupies a privileged position: it is the unique positive-definite even unimodular lattice in 8 dimensions, arises from octonionic constructions, and contains the Standard Model gauge groups as subgroups. Geometrically, its root system is constructed from two interleaved copies of the H_4 root system scaled by the golden ratio $\phi = (1 + \sqrt{5})/2$.

These connections suggest a deep structural unity. We formalize this unity pre-dynamically via a categorical framework that extracts stable algebraic invariants under recursive lifts along the exceptional chain.

2 The Exceptional-Octonionic Hierarchy

The structures are linked octonionically as follows:

$$G_2 \cong \text{Aut}(\mathbb{O}), \text{ automorphisms of the octonions.} \quad (1)$$

$$F_4 \cong \text{Aut}(J_3(\mathbb{O})), \text{ automorphisms of the 27-dimensional exceptional Jordan algebra.} \quad (2)$$

$$E_6 \cong \text{Der}(J_3(\mathbb{O})), \text{ derivations preserving the cubic norm.} \quad (3)$$

$$E_7, E_8 \text{ arise from the Freudenthal triple system over } J_3(\mathbb{O}). \quad (4)$$

Maximal regular embeddings include $E_8 \supset SU(3) \times E_6$ and $E_8 \supset SU(2) \times E_7$, with chain extensions $E_7 \supset E_6 \times U(1)$ and $E_6 \supset SU(3)^3 \times U(1)$ (trinification). Geometrically: $\Lambda_{E_8} \simeq H_4 \cup \phi^{-1}H_4$ via even icosians.

This forms a maximal unifying chain (unique up to isomorphism of maximal regular embeddings), terminating at E_8 , with G_2 and F_4 as structural intermediaries.

3 The Category ExcOct and the RIF Functor

3.1 Definition

Define the category **ExcOct**:

- **Objects:** Pairs $(\mathcal{G}, \Sigma)$ for $\mathcal{G} \in \{U(1), SU(2), SU(3), H_4, G_2, F_4, E_6, E_7, E_8\}$, where objects include compact Lie groups and the finite Coxeter group equipped with compatible root or reflection data, and Σ is the associated algebraic or geometric structure (e.g., root systems, Jordan algebras, Freudenthal triple systems).
- **Morphisms:** Admissible lifts—maximal regular embeddings, derivation closures, golden-ratio projections, and trinification branchings.

The primary chain has terminal object E_8 .

3.2 Definition (RIF Functor)

Define the Recursion-Invariant Functor (RIF) functor

$$\mathcal{R} : \mathbf{ExcOct} \longrightarrow \mathbf{Inv},$$

where **Inv** is the category of sets of algebraic invariants (polynomials, traces, norms, etc.) with morphisms given by restriction.

For each object $(\mathcal{G}, \Sigma)$,

$$\mathcal{R}(\mathcal{G}, \Sigma) = \bigcap_{f: (\mathcal{G}, \Sigma) \rightarrow (\mathcal{G}', \Sigma')} f^*(\text{Inv}(\mathcal{G}')) \cap \text{Fix}(\text{Aut}(\mathcal{G}')).$$

3.2.1 Definition 3.1 (Fundamental Algebraic Invariant)

For any object $(\mathcal{G}, \Sigma)$ in **ExcOct**, a polynomial $P \in \mathcal{R}(\mathcal{G}, \Sigma)$ is **fundamental** if and only if:

1. $P \notin \mathcal{R}(\mathcal{G}', \Sigma')$ for every proper subobject $(\mathcal{G}', \Sigma') \subset (\mathcal{G}, \Sigma)$;
2. $\dim(\text{Stab}_P(\text{Aut}(\mathcal{G}))) < \dim(\text{Aut}(\mathcal{G}))$;
3. P generates a maximal proper subalgebra of $\text{Inv}(\mathcal{G})$.

Equivalently, P is an *irreducible fixed point* of the RIF functor, representing independent algebraic order parameters in the unification chain.

3.3 Proposition 3.1 (Covariance)

For any morphism $f : (\mathcal{G}, \Sigma) \rightarrow (\mathcal{G}', \Sigma')$,

$$\mathcal{R}(f) : \mathcal{R}(\mathcal{G}, \Sigma) \rightarrow \mathcal{R}(\mathcal{G}', \Sigma')$$

is well-defined. Covariance follows from the closure of admissible morphisms under composition and the pullback identity $(g \circ f)^* = f^* \circ g^*$.

3.4 Proposition 3.2 (Idempotence)

For all $(\mathcal{G}, \Sigma)$,

$$\mathcal{R}^2(\mathcal{G}, \Sigma) = \mathcal{R}(\mathcal{G}, \Sigma).$$

Proof. A second stabilization re-applies the same set of enclosing pullbacks and automorphism fixations that defined the first, thus producing no further refinement. $\square$ $\square$

3.5 Proposition 3.3 (Full Functoriality / Naturality)

The RIF operator $\mathcal{R} : \mathbf{ExcOct} \rightarrow \mathbf{Inv}$ is a covariant functor with natural inclusions on morphisms.

Proof. Let $f : (\mathcal{G}, \Sigma) \rightarrow (\mathcal{G}', \Sigma')$ be any morphism, and $P \in \mathcal{R}(\mathcal{G}, \Sigma)$.

For any outgoing morphism $k : \mathcal{G}' \rightarrow \mathcal{G}''$, the composition $k \circ f$ is outgoing from $\mathcal{G}$. Then

$$P \in (k \circ f)^*(\text{Inv}(\mathcal{G}'')) = k^*(f^*(\text{Inv}(\mathcal{G}''))),$$

so $f^*(P) \in k^*(\text{Inv}(\mathcal{G}''))$ for every k .

Hence, $f^*(P) \in \mathcal{R}(\mathcal{G}', \Sigma')$.

The induced map $\mathcal{R}(f)$ is thus a natural inclusion sending stabilized invariants to stabilized invariants. Pullback compatibility ensures

$$\mathcal{R}(g \circ f) = \mathcal{R}(g) \circ \mathcal{R}(f), \quad \mathcal{R}(\text{id}) = \text{id},$$

so $\mathcal{R}$ is a well-defined covariant functor. $\square$ $\square$

3.6 Corollary 3.4 (Consistency)

Together with established covariance, idempotence, and naturality results,

$$\boxed{\mathcal{R}^2 = \mathcal{R}, \quad \mathcal{R}(g \circ f) = \mathcal{R}(g) \circ \mathcal{R}(f)}$$

the RIF operator is a fully covariant, idempotent functor that extracts all recursion-stable invariants along the exceptional-octonionic hierarchy. This completes the categorical foundation of the RIF framework. The operator acts as a closure under symmetry-preserving recursion, identifying the seven invariant algebraic structures as forced fixed points of the exceptional chain.

4 Theorem: Forced Emergent Algebraic Invariants

4.1 Theorem 4.1

The RIF functor along the primary unifying chain yields exactly seven non-trivial fundamental algebraic invariants.

Proof. See explicit computation in Section 5. □ □

4.2 Corollary 4.2 (Exhaustiveness)

No further independent invariants arise within finite-dimensional simple exceptional structures. G_2 and F_4 mediate octonionic and Jordan structure but their invariants are subsumed under higher stabilization (E_6 and above).

5 Explicit Computation and Completeness

The RIF functor yields exactly seven fundamental invariants along the chain $H_4 \rightarrow E_8 \supset E_7 \times SU(2) \supset E_6 \times U(1) \supset SU(3)^3 \times U(1)$:

$\mathcal{G}$	Primary Invariant	Order	$\dim \mathcal{R}(\mathcal{G})$
$U(1)$	Phase mod 2π	1	1
$SU(2)$	$\mathbb{Z}_2$ center	2	1
$SU(3)$	Triadic closure (f^{abc})	3	1
H_4	Golden-ratio ϕ	irrational	1
E_6	Cubic norm $J_3(\mathbb{O})$	3	1
E_7	Quartic FTS invariant	4	1
E_8	Derivation closure	structural	1
G_2, F_4	Subsumed	–	0

Lemma 5.1 (Casimir Collapse). *Higher Casimirs destabilize under RIF lifts. For E_6 (6 Casimirs: degrees 2,8,12,14,18,24), only the cubic norm N_3 is newly fundamental at E_6 (the*

degree-2 form is inherited from subobjects); others couple to coset degrees under $E_6 \rightarrow E_7$. Similarly for E_7 (7 Casimirs) and E_8 (8 Casimirs): $\mathcal{R}$ extracts only the invariants satisfying Definition 3.1—those non-inherited, non-generic, and generating maximal subalgebras. Since $\mathcal{R}^2 = \mathcal{R}$ (idempotence) and each lift f^* restricts invariants monotonically, no additional independent generators emerge. Thus exactly seven fundamental invariants span the stabilized ring.

6 Relation to the Standard Model Gauge Group

The hierarchy naturally embeds the gauge-algebra structure underlying the Standard Model:

$$E_8 \supset SU(3) \times E_6, \quad E_8 \supset SU(2) \times E_7, \quad E_7 \supset E_6 \times U(1).$$

Charge quantization follows from the discrete spectrum of normalized $U(1)$ Cartan generators in the maximal regular embeddings. Chirality, family replication, couplings, and symmetry breaking require additional dynamical input outside this pre-dynamical framework.

7 Discussion

The RIF framework extracts a complete set of seven pre-dynamical algebraic invariants from the exceptional-octonionic hierarchy, forced by symmetry, recursion stabilization, and termination at E_8 . The golden-ratio geometry of H_4 and the octonionic foundation (mediated by G_2 and F_4) provide the base from which continuous Lie symmetries emerge.

This synthesis reconciles discrete icosahedral symmetry with continuous exceptional Lie structure in a purely algebraic, recursion-invariant setting.

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